INFLUENCE OF THE BAUSCHINGER EFFECT ON THE BUCKLING OF A COMPRESSED STRIP

(OB UCHETE VLIIANIIA EFFEKTA BAUSHINGERA NA POTERIU USTOICHIVOSTI SZHATOI POLOSY)

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In investigations of the buckling of bars beyond the elastic range by von Karman [1] and by Shanley [2], one is required to know only the material properties at a given value of the critical force; the latter may depend upon the tangent and secant moduli at a given point on the σ, ε curve and on the modulus of elasticity (and, it goes without saying, also on the bar geometry). Thus the Bauschinger effect, which appears as a consequence of the acquired anisotropy of the material, is not taken into account. This fact suggests approximations in the work of von Karman and Shanley. The present paper, following an idea of Leibenzon [3] and Ishlinskii [4] considers the problem of the buckling of a uniformly compressed strip in the case of plane strain.

The Shanley representation [2] is used.

It is shown that for other conditions being equal, the critical force does not depend on the nature of the acquired anisotropy.

We consider a rectangular strip with sides a and b, compressed by a uniformly distributed load of intensity p (Fig. 1). We employ the plasticity relations of Prager [5] and of Ishlinskii [6]. We take the plasticity condition in the form [7]

$$[(\sigma_x - c\varepsilon_{[x]}) - (\sigma_y - c\varepsilon_{[y]})]^2 + 4(\tau_{xy} - c\varepsilon_{[xy]})^2 = 4f^2(\Gamma) \quad (1)$$

$$\Gamma = \frac{1}{4}(\varepsilon_{[x]} - \varepsilon_{[y]})^2 + \varepsilon_{[xy]}{}^2$$

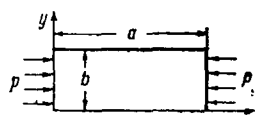

Fig. 1.

where σ_{ij} are the stress components, ε_{ij} the strain components.

Here and in the future, square brackets in the indices denote values relating to the plastic range and curved brackets to the elastic range; they are omitted where not required.

The associated relations of plastic flow take the form

$$\frac{d\varepsilon_{[x]}}{(\sigma_x - c\varepsilon_{[x]}) - (\sigma_y - c\varepsilon_{[y]})} = \frac{d\varepsilon_{[y]}}{(\sigma_y - c\varepsilon_{[y]}) - (\sigma_x - c\varepsilon_{[x]})} = \frac{d\varepsilon_{[xy]}}{2(\tau_{xy} - c\varepsilon_{[xy]})} \qquad (2)$$

The total strain is the sum of its components

$$\varepsilon_{ij} = \varepsilon_{(ij)} + \varepsilon_{[ij]} \qquad (3)$$

For simplicity we shall consider the material to be incompressible.

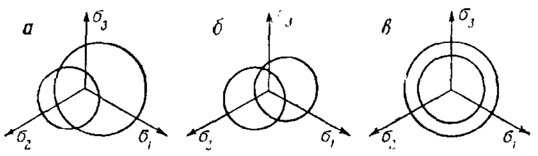

Fig. 2.

Relations (1) and (2) in the general case determine the behavior of an anisotropic strain hardening material for which the flow surface displaces and widens (Fig. 2*a*). In the case f = const the flow surface moves as a rigid whole (Fig. 2*b*). In the case $c = 0$ the strain hardening is isotropic and the flow surface varies in the same way (Fig. 2*c*).

By limiting ourselves to the case of linear strain hardening, the plasticity condition (1) takes the form

$$\sqrt{[(\sigma_x - c\varepsilon_{[x]}) - (\sigma_y - c\varepsilon_{[y]})]^2 + 4(\tau_{xy} - c\varepsilon_{[xy]})^2} = 2k + d\cdot\sqrt{(\varepsilon_{[x]} - \varepsilon_{[y]})^2 + 4\varepsilon_{[xy]}^2} \qquad (4)$$

where c, d and k are constants.

For a strip compressed into the plastic range, the relations

$$\sigma_x = -p, \qquad \sigma_y = \tau_{xy} = 0, \qquad \varepsilon_x + \varepsilon_y = 0, \qquad \varepsilon_{xy} = 0 \qquad (5)$$

hold up to the buckling load.

The relation between the compressive load p and the compressive strain ε_x is obtained from (3), (4) and (5):

$$-p\left(1+\frac{c+d}{G}\right)=2k+2(c+d)\varepsilon_x \tag{6}$$

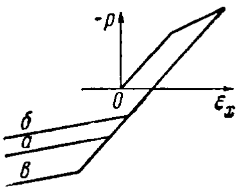

Fig. 3.

Consequently, if the quantity $c + d$ is a certain fixed constant q (tangent modulus), then the nature of the relation of $-p$ to ε_x is one and the same for any plasticity condition (4); i.e. compression experiments do not permit the determination of which plasticity condition (4) holds.

Figure 3 shows the relation of $-p$ to ε_x for the flow conditions of Fig. 2.

It is clear that the critical force will either depend upon the value of q alone or also upon it and the values of c and d.

Assume that buckling of the strip occurs for $p = p^*$.

We seek a solution of the problem in the form

$$\sigma_x=\sigma_x^0+\sigma_x',\ldots,\varepsilon_x=\varepsilon_x^0+\varepsilon_x',\ldots,u=u^0+u',\ldots \tag{7}$$

The components with zero superscripts determine the state of stress and strain up to the buckling load and are given by Formulas (5) and (6). The components with primed superscripts are those introduced by the buckling. The problem reduces to the determination of these components.

The equation of equilibrium has the form

$$\frac{\partial\sigma_x'}{\partial x}+\frac{\partial\tau_{xy}'}{\partial y}=0,\qquad \frac{\partial\tau_{xy}'}{\partial x}+\frac{\partial\sigma_y'}{\partial y}=0 \tag{8}$$

The increments $\delta\varepsilon_{[ij]}$ appear in the relations of the plasticity law, Equation (2). During buckling the components of induced strain are small and coincide with the strain increments, so that one must take

$$d\varepsilon_{[ij]}\approx\varepsilon_{[ij]}$$

By linearization of relations (2) and (4) for plastic strain components we shall have

$$\sigma_x'-\sigma_y'=2q\varepsilon_{[x]}',\qquad \varepsilon_{[xy]}'=0 \tag{9}$$

By use of the relations of the law of elastic strain from (9) and by taking account of (3), we obtain

$$\sigma_x'-\sigma_y'=\frac{2q}{[1+q/G]}\,\varepsilon_x',\qquad \tau_{xy}'=2G\varepsilon_{xy}' \tag{10}$$

For the induced state there are the boundary conditions

$$\sigma_y' = 0, \qquad \tau_{xy}' + p\frac{\partial v'}{\partial x} = 0 \quad \text{when } y = 0,\ y = b \tag{11}$$

By assuming that $u' = \partial\Phi/\partial y$, $v' = -\partial\Phi/\partial x$, we satisfy the equation of incompressibility. For determination of the function $\Phi(x, y)$ from (8) and (10), taking account of the relations between strains and displacements, we obtain the equation

$$\frac{\partial^4\Phi}{\partial x^4} - 2\beta^2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4} = 0 \qquad \left(\beta^2 = \frac{1 - q/2G}{1 + q/2G}\right) \tag{12}$$

The general integral of Equation (12) has the form

$$\Phi(x, y) = [C_1Y_1(y) + C_2Y_2(y) + C_3Y_3(y) + C_4Y_4(y)]\cos nx \tag{13}$$

Here

$$Y_1(y) = \cosh\alpha_1 y\sin\alpha_2 y, \qquad Y_2(y) = \cosh\alpha_1 y\cos\alpha_2 y$$
$$Y_3(y) = \sinh\alpha_1 y\cos\alpha_2 y, \qquad Y_4(y) = \sinh\alpha_1 y\sin\alpha_2 y$$

where $\alpha_1 = n\sqrt{[(1 - \beta^2)/2]}$, $\alpha_2 = n\sqrt{[(1 + \beta^2)/2]}$, C_i ($i = 1, 2, 3, 4$) are arbitrary constants of integration.

By expressing the stress and displacement components in terms of the function (13) and substitution into the boundary conditions (11), we obtain a homogeneous linear system of algebraic equations in the C_i.

During buckling $C_i \neq 0$, therefore the determinant of the system is equal to zero. From this we have the equation giving the value of the critical force

$$\gamma = \sqrt{1 - \beta^2}\,\frac{(\alpha_2 b)\sinh\alpha_1 b - (\alpha_1 b)\sin\alpha_2 b}{(\alpha_2 b)\sin\alpha_2 b} \qquad (\gamma = p^*/G) \tag{14}$$

At the edges of the strip we take $n = m\pi/a$ in order to satisfy the geometrical conditions; m is the number of half waves ($m \geqslant 1$). By taking the width of the strip sufficiently small and by expanding the right-hand side of (14) in series we obtain the formula

$$\gamma = \frac{1 - \beta^2}{6}\,\frac{m^2\pi^2b^2}{a^2} \tag{15}$$

in which second order terms have been retained.

Consider the ratio $\eta = [\gamma]/(\gamma)$, where $[\gamma]$ is known from (15) and (γ) from the Bryan formula for elastic buckling of a strip. Then it follows from (15) that

$$\eta = \frac{q}{2G + q}$$

Thus, the critical force depends only on the constants G and q (which also follows from the buckling theory for bars).

The parameter c, characterizing the effect of the acquired anisotropy (Bauschinger effect) does not appear in the expression for the critical force.

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BIBLIOGRAPHY

1. Von Karman, T., Untersuchungen über Knickfestigkeit Mitteilungen über Forschungsarbeiten auf dem Gebiete des Ingenieurwesens. Heft 81, Berlin, 1910.

2. Shanley, F.R., The column paradox. *J. Aero. Sci.* Vol. 13, 1946.

3. Leibenzon, L.S., O primenenii garmonicheskikh funktsii k voprosy ob ustoichivosti sfericheskoi i tsilindricheskoi obolochek (On the application of harmonic functions to problems on the stability of spherical and cylindrical shells). *Sobr. tr.* Vol. 1, 1951.

4. Ishlinskii, A.Iu., Rassmotrenie voprosov ob ustoichivosti ravnovesiia uprugikh tel s tochki zreniia matematicheskoi teorii uprugosti (Consideration of problems in the stability equilibrium of bodies from the point of view of the mathematical theory of elasticity). *Ukr. matem. zhurnal* Vol. 6, No. 2, 1954.

5. Prager, W., Der Einfluss der Verformung auf die Fliessbedingung Zähplastischer Körper. *ZAMM* Bd. 15, s. 76-80, 1935.

6. Ishlinskii, A.Iu., Obshchaia teoriia plastichnosti s lineinym uprochneniem (General theory of plasticity with linear strain hardening). *Ukr. matem. zhurnal* Vol. 6, No. 3, 1954.

7. Ivlev, D.D., K teorii ploskoi deformatsii uprochniaiushchegosia plasticheskogo materiala (Theory of plane strain for strain-hardening plastic material). *PMM* Vol. 24, No. 4, 1960.

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